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WEAK CONVERGENCE THEOREMS IN FEYNMAN'S OPERATIONAL CALCULI : THE CASE OF TIME DEPENDENT NONCOMMUTING OPERATORS

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Abstract. Feynman's operational calculus for noncommuting operators was studied by means of measures on the time inteval. And various stability theorems for Feynman's operational calculus were investigated. In this paper we see the time-dependent stability properties when the operator-valued functions take their values in a separable Hilbert space.

1. Introduction

Feynman's operational calculus originated with the 1951 paper [5] and concerns with the formation of functions for noncommuting operators. Much work on this subject has been done by mathematicians and physicsts. Refernces can be found in the books of Johnson and Lapidus [11] and Nazaikinskii, Shatalov and Sternin [14]. The setting of the operational calculus used in this paper is that developed by Jefferies and Johnson in the papers [6]-[10]. The Jefferies-Johnson approach to the operational calculus uses measures on the time interval $[0, T]$ to determine the order of operators in products.

We now introduce some notations and begin to our discussion more precise. Let X be a separable Hilbert space over the complex numbers and let $\mathcal{L}(X)$ denote the space of bounded linear operators on X. Fix $T > 0$. For $i = 1, \cdots, n$ let $A_i : [0, T] \to \mathcal{L}(X)$ be maps that are measurable in the sense that $A_i^{-1}(E)$ is a Borel set in [0,T] for any strong operator open set $E \subset \mathcal{L}(X)$. To each $A_i(\cdot)$ we associate a finite

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Borel measure λ_i on $[0,T]$ and we require that, for each i,

$$r_i = \int_{[0,T]} ||A_i(s)||_{\mathcal{L}(X)} |\lambda_i|(ds) < \infty.$$

For n positive numbers $r_1, \cdots, r_n$, let $\mathbb{A}(r_1, \cdots, r_n)$ be the space of complex-valued functions of n complex variables $f(z_1, \cdots, z_n)$, which are analytic at the origin and are such that their power series expansion

$$f(z_1, \cdots, z_n) = \sum_{m_1, \cdots, m_n = 0}^{\infty} c_{m_1, \cdots, m_n} z_1^{m_1} \cdots z_n^{m_n}$$

converges absolutely at least on the closed polydisk $|z_1| \leq r_1, \cdots, |z_n| \leq r_n$. Such functions are analytic at least in the open polydisk $|z_1| < r_1, \cdots, |z_n| < r_n$.

To the algebra $\mathbb{A}(r_1, \cdots, r_n)$ we associate as in [6] a disentangling algebra by replacing the z_i's with formal commuting objects $(A_i(\cdot), \lambda_i)^{\sim}$, $i = 1, \cdots, n$. Rather than using the notation $(A_i(\cdot), \lambda_i)^{\sim}$ below, we will often abbreviate to $A_i(\cdot)^{\sim}$. Consider the collection $\mathbb{D}((A_1(\cdot), \lambda_1)^{\sim}, \cdots, (A_n(\cdot), \lambda_n)^{\sim})$ of all expressions of the form

$$f(A_1(\cdot)^{\sim}, \cdots, A_n(\cdot)^{\sim}) = \sum_{m_1, \cdots, m_n = 0}^{\infty} c_{m_1, \cdots, m_n} (A_1(\cdot)^{\sim})^{m_1} \cdots (A_n(\cdot)^{\sim})^{m_n}$$

where $c_{m_1, \cdots, m_n} \in \mathbb{C}$ for all $m_1, \cdots, m_n = 0, 1, \cdots$, and

$$\begin{aligned} ||f(A_1(\cdot)^{\sim}, \cdots, \; A_n(\cdot)^{\sim})|| &= ||f(A_1(\cdot)^{\sim}, \cdots, A_n(\cdot)^{\sim})||_{\mathbb{D}(A_1(\cdot)^{\sim}, \cdots, A_n(\cdot)^{\sim})} \\ &= \textstyle\sum_{m_1, \cdots, m_n = 0}^{\infty} |c_{m_1, \cdots, m_n}| r_1^{m_1} \cdots r_n^{m_n} < \infty. \end{aligned} \tag{1}$$

The function on $\mathbb{D}((A_1(\cdot), \lambda_1)^{\sim}, \cdots, (A_n(\cdot), \lambda_n)^{\sim})$ defined by (1) makes $\mathbb{D}((A_1(\cdot), \lambda_1)^{\sim}, \cdots, (A_n(\cdot), \lambda_n)^{\sim})$ into a commutative Banach algebra [10].

We refer to $\mathbb{D}((A_1(\cdot), \lambda_1)^{\sim}, \cdots, (A_n(\cdot), \lambda_n)^{\sim})$ as the disentangling algebra associated with the n-tuple $((A_1(\cdot), \lambda_1)^{\sim}, \cdots, (A_n(\cdot), \lambda_n)^{\sim})$. We will often write $\mathbb{D}$ in place of $\mathbb{D}(A_1(\cdot)^{\sim}, \cdots, A_n(\cdot)^{\sim})$ or $\mathbb{D}((A_1(\cdot), \lambda_1)^{\sim}, \cdots, (A_n(\cdot), \lambda_n)^{\sim})$.

For $m = 0, 1, \cdots$, let S_m denote the set of all permutations of the integers $\{1, \cdots, m\}$, and given $\pi \in S_m$, we let

$$\Delta_m(\pi) = \{(s_1, \cdots, s_m) \in [0,T]^m : 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < T\}.$$

Now for nonnegative integers $m_1, \cdots, m_n$ and $m = m_1 + \cdots + m_n$, we define

$$C_i(s) = \begin{cases} A_1(s), & if\ i \in \{1, \cdots, m_1\} \\ A_2(s), & if\ i \in \{m_1+1, \cdots, m_1+m_2\} \\ \ \vdots & \\ A_n(s), & if\ i \in \{m_1+\cdots+m_{n-1}+1, \cdots, m\} \end{cases}$$

for $i = 1, \cdots, m$ and for all $0 \le s \le T$. Next, in order to accommodate the use of discrete measures, we will need a refined version of the time-ordered sets $\Delta_m(\pi)$. Let $\tau_1, \cdots, \tau_h \in [0, T]$ be such that $0 < \tau_1 < \cdots < \tau_h < T$. Given $m \in \mathbb{N}$ and $\pi \in S_m$, and nonnegative integers $r_1, \cdots, r_{h+1}$ such that $r_1 + \cdots + r_{h+1} = m$, we define

$$\Delta_{m;r_1,\cdots,r_{h+1}}(\pi) = \{(s_1, \cdots, s_m) \in [0,T]^m : 0 < s_{\pi(1)} < \cdots < s_{\pi(r_1)} \\ < \tau_1 < s_{\pi(r_1+1)} < \cdots < s_{\pi(r_1+r_2)} < \tau_2 < s_{\pi(r_1+r_2+1)} < \cdots < \\ s_{\pi(r_1+\cdots+r_h)} < \tau_h < s_{\pi(r_1+\cdots+r_h+1)} < \cdots < s_{\pi(m)} < T\}.$$

Now let $\lambda_1, \cdots, \lambda_n$ be finite Borel measures on $[0, T]$ such that

$$\lambda_l = \mu_l + \eta_l$$

for $l = 1, \cdots, n$ where μ_l is a continuous measure and η_l is a finitely supported discrete measure for each l. Let $\{\tau_1, \cdots, \tau_h\}$ be the set obtained by taking the union of the supports of the discrete measures $\eta_1, \cdots, \eta_n$ and write

$$\eta_l = \sum_{i=1}^{h} p_{li}\delta_{\tau_i}$$

for each $l = 1, \cdots, n$. With this notation it may be that many of the $p_{li}'s$ are equal to zero. Now we define the disentangling map $\mathcal{T}_{\lambda_1,\cdots,\lambda_n}$ which will take us from the commutative framework of the disentamgling algebra $\mathbb{D}$ $(\tilde{A_1(\cdot)}, \cdots, \tilde{A_n(\cdot)})$ to the noncommutative setting of $\mathcal{L}(X)$.

Definition 1.1. *Let $P^{m_1,\cdots,m_n}(z_1, \cdots, z_n) = z_1^{m_1} \cdots z_n^{m_n}$. We define the disentangling map on this monomial by*

$$\mathcal{T}_{\lambda_1,\cdots,\lambda_n} P^{m_1,\cdots,m_n}(\tilde{A_1(\cdot)}, \cdots, \tilde{A_n(\cdot)}) \\ = \mathcal{T}_{\lambda_1,\cdots,\lambda_n}((\tilde{A_1(\cdot)})^{m_1} \cdots (\tilde{A_n(\cdot)})^{m_n})$$

$$
\begin{aligned}
&:= \sum_{q_{11}+q_{12}=m_1} \sum_{q_{21}+q_{22}=m_2} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left(\frac{m_1! \cdots m_n!}{q_{11}!q_{12}!q_{21}!q_{22}! \cdots q_{n1}!q_{n2}} \right) \\
&\sum_{\pi \in S_{q_{11}+q_{21}+\cdots+q_{n1}}} \sum_{r_1+\cdots+r_{h+1}=q_{11}+q_{21}+\cdots+q_{n1}} \\
&\sum_{j_{11}+\cdots+j_{1h}=q_{12}} \sum_{j_{21}+\cdots+j_{2h}=q_{22}} \cdots \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \\
&\left(\frac{q_{12}!q_{22}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! j_{21}! \cdots j_{2h}! j_{n1}! \cdots j_{nh}!} \right) \int_{\Delta_{q_{11}+q_{21}+\cdots+q_{n1}; r_1, \cdots, r_{h+1}}(\pi)} \\
&C_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}) \cdots \\
&C_{\pi(r_1+\cdots+r_h+1)}(s_{\pi(r_1+\cdots+r_h+1)})[p_{nh}A_n(\tau_h)]^{j_{nh}} \cdots [p_{2h}A_2(\tau_h)]^{j_{2h}} \\
&[p_{1h}A_1(\tau_h)]^{j_{1h}} C_{\pi(r_1+\cdots+r_h)}(s_{\pi(r_1+\cdots+r_h)}) \cdots C_{\pi(r_1+1)}(s_{\pi(r_1+1)}) \\
&[p_{n1}A_n(\tau_1)]^{j_{n1}} \cdots [p_{21}A_2(\tau_1)]^{j_{21}} [p_{11}A_1(\tau_1)]^{j_{11}} C_{\pi(r_1)}(s_{\pi(r_1)}) \cdots \\
&C_{\pi(1)}(s_{\pi(1)})(\mu_1^{q_{11}} \times \cdots \times \mu_n^{q_{n1}})(ds_1, \cdots, ds_{q_{11}+q_{21}+\cdots+q_{n1}}).
\end{aligned}
$$

Finally for $f \in \mathbb{D}((A_1(\cdot), \lambda_1)\tilde{}, \cdots, (A_n(\cdot), \lambda_n)\tilde{})$ *given by*

$$
f(A_1(\cdot)\tilde{}, \cdots, A_n(\cdot)\tilde{}) = \sum_{m_1, \cdots, m_n=0}^{\infty} c_{m_1, \cdots, m_n} (A_1(\cdot)\tilde{})^{m_1} \cdots (A_n(\cdot)\tilde{})^{m_n}
$$

we set

$$
\begin{aligned}
&\mathcal{T}_{\lambda_1, \cdots, \lambda_n} f(A_1(\cdot)\tilde{}, \cdots, A_n(\cdot)\tilde{}) \\
&:= \sum_{m_1, \cdots, m_n=0}^{\infty} c_{m_1, \cdots, m_n} \mathcal{T}_{\lambda_1, \cdots, \lambda_n} P^{m_1, \cdots, m_n}(A_1(\cdot)\tilde{}, \cdots, A_n(\cdot)\tilde{}).
\end{aligned}
$$

We will often use the alternate notation indicated in the next two equalities :

$$
P^{m_1, \cdots, m_n}_{\lambda_1, \cdots, \lambda_n}(A_1(\cdot), \cdots, A_n(\cdot)) = \mathcal{T}_{\lambda_1, \cdots, \lambda_n} P^{m_1, \cdots, m_n}(A_1(\cdot)\tilde{}, \cdots, A_n(\cdot)\tilde{})
$$

and

$$
f_{\lambda_1, \cdots, \lambda_n}(A_1(\cdot), \cdots, A_n(\cdot)) = \mathcal{T}_{\lambda_1, \cdots, \lambda_n} f(A_1(\cdot)\tilde{}, \cdots, A_n(\cdot)\tilde{}).
$$

2. Stability properties

Let S be a metric space and let $\{\lambda_k\}_{k=1}^{\infty}$ be a sequence of finite Borel measures on S. We say that this sequence converges weakly to a finite Borel measure λ on S and write $\lambda_k \rightharpoonup \lambda$ if for every bounded continuous real-valued function f on S we have $\int_S f(s)\, \lambda_k(ds) \to \int_S f(s)\, \lambda(ds)$ as

$k \to \infty$. The following results are Lemma 3.1 of [12] and Theorem 2.4 of [13].

LEMMA 2.1. *Let* $\eta = \sum_{i=1}^{h} p_i \delta_{\tau_i}$ *be a purely discrete probability measure on* $[0, T]$ *with finite support. Assume that* $0 < \tau_1 < \cdots < \tau_h < T$. *Let*

$$\alpha_i = \min\{\tau_i - \tau_{i-1}, \tau_{i+1} - \tau_i\}$$

for $i = 1, \cdots, h$ *where we take* $\tau_0 = 0$ *and* $\tau_{h+1} = T$. *In each interval* $(\tau_i - \alpha_i, \tau_i + \alpha_i), i = 1, \cdots, h$ *choose sequences* $\{\tau_{ik}\}_{k=1}^{\infty}$. *For each* $i = 1, \cdots, h$ *choose a sequence* $\{p_{ik}\}_{k=1}^{\infty}$ *such that* $\eta_k = \sum_{i=1}^{h} p_{ik} \delta_{\tau_{ik}}$ *is a probability measure for each* k. *Then* $\eta_k \rightharpoonup \eta$ *if and only if*

$$\begin{cases} p_{ik} \to p_i \quad \text{and} \quad \tau_{ik} \to \tau_i & \text{if } p_i \neq 0, \\ p_{ik} \to p_i \quad \text{and} \quad \{\tau_{ik}\}_{k=1}^{\infty} \text{bounded} & \text{if } p_i = 0 \end{cases}$$

for $i = 1, \cdots, h$.

LEMMA 2.2. *Let* X *be a separable Hilbert space. Let* μ_k, μ *be Borel probability measures on the metrisc space* S *for* $k \in \mathbb{N}$. *Let* $f_k, f, k \in \mathbb{N}$, *be continuous norm bounded* X*-valued functions on* S. *If* $\mu_k \rightharpoonup \mu$ *and if* $f_k \to f$ *uniformly in* X*-norm on* S, *then*

$$\lim_{k \to \infty} \int_E f_k d\mu_k = \int_E f d\mu$$

in norm for any Borel set $E \subset S$ *with* $\mu(\partial E) = 0$.

First we consider the disentangling map for $P^{m_1, \cdots, m_n}(A_1(\cdot)\tilde{)}, \cdots, A_n(\cdot)\tilde{)})$.

THEOREM 2.3. *Let* $A_l : [0, T] \to \mathcal{L}(X), X$ *a separable Hilbert space, be continuous with respect to the norm topology on* $\mathcal{L}(X)$ *for each* $l = 1, 2, \cdots, n$. *And let* $\lambda_1, \cdots, \lambda_n$ *be finite Borel measures on* $[0, T]$ *such that*

$$\lambda_l = \mu_l + \eta_l$$

for $l = 1, \cdots, n$ *where* μ_l *is a continuous probability measure and* η_l *is a finitely supported discrete probability measure for each* l. *Let* $\{\tau_1, \cdots, \tau_h\}$ *be the set obtained by taking the union of the supports of the discrete measures* $\eta_1, \cdots, \eta_n$ *and write*

$$\eta_l = \sum_{i=1}^{h} p_{li} \delta_{\tau_i}$$

for each $l = 1, \cdots, n$. *Choose sequences* $\{\mu_{lk}\}_{k=1}^{\infty}, l = 1, \cdots, n$ *of continuous Borel probability measures on* $[0, T]$ *such that* $\mu_{lk} \rightharpoonup \mu_l$. *Also* *choose sequences* $\{\eta_{lk}\}_{k=1}^{\infty}, l = 1, \cdots, n$ *of discrete probability measures on* $[0,T]$ *as in Lemma 2.1 such that* $\eta_{lk} \rightharpoonup \eta_l$*; i.e. write*

$$\eta_{lk} = \sum_{i=1}^{h} p_{li}^k \delta_{\tau_{ik}},$$

where, as in the Lemma 1, $p_{li}^k \to p_{li}$ *and* $\tau_{ik} \to \tau_i$ *as* $k \to \infty$ *for all* i, l *assuming that for* $p_{li} \neq 0$ *for all* i, l*. Finally let* $\lambda_{lk} = \mu_{lk} + \eta_{lk}$ *for* $l = 1, \cdots, n$*. Then for any nonnegative integers* $m_1, \cdots, m_n$ *and for any* $\phi \in X$

$$\lim_{k\to\infty} P_{\lambda_{1k},\cdots,\lambda_{nk}}^{m_1,\cdots,m_n}(A_1(\cdot), \cdots, A_n(\cdot))\phi$$
$$= P_{\lambda_1,\cdots,\lambda_n}^{m_1,\cdots,m_n}(A_1(\cdot), \cdots, A_n(\cdot))\phi.$$

Proof. We see that for any $\phi \in X$

$$||P_{\lambda_{1k},\cdots,\lambda_{nk}}^{m_1,\cdots,m_n}(A_1(\cdot), \cdots, A_n(\cdot))\phi \quad - P_{\lambda_1,\cdots,\lambda_n}^{m_1,\cdots,m_n}(A_1(\cdot), \cdots, A_n(\cdot))\phi||$$
$$\leq \sum_{q_{11}+q_{12}=m_1} \sum_{q_{21}+q_{22}=m_2} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left(\frac{m_1! \cdots m_n!}{q_{11}!q_{12}!q_{21}!q_{22}! \cdots q_{n1}!q_{n2}!} \right)$$
$$\sum_{\pi \in S_{q_{11}+q_{21}+\cdots+q_{n1}}} \sum_{r_1+\cdots+r_{h+1}=q_{11}+q_{21}+\cdots+q_{n1}}$$
$$\sum_{j_{11}+\cdots+j_{1h}=q_{12}} \sum_{j_{21}+\cdots+j_{2h}=q_{22}} \cdots \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}}$$
$$\left(\frac{q_{12}!q_{22}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}!j_{21}! \cdots j_{2h}! \cdots j_{n1}! \cdots j_{nh}!} \right) || \int_{\Delta_{q_{11}+q_{21}+\cdots+q_{n1};r_1,\cdots,r_{h+1}}(\pi)}$$
$$C_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}) \cdots$$
$$C_{\pi(r_1+\cdots+r_h+1)}(s_{\pi(r_1+\cdots+r_h+1)})[p_{nh}^k A_n(\tau_{hk})]^{j_{nh}} \cdots [p_{2h}^k A_2(\tau_{hk})]^{j_{2h}}$$

$$[p_{1h}^k A_1(\tau_{hk})]^{j_{1h}} C_{\pi(r_1+\cdots+r_h)}(s_{\pi(r_1+\cdots+r_h)}) \cdots C_{\pi(r_1+1)}(s_{\pi(r_1+1)})$$
$$[p_{n1}^k A_n(\tau_{1k})]^{j_{n1}} \cdots [p_{21}^k A_2(\tau_{1k})]^{j_{21}} [p_{11}^k A_1(\tau_{1k})]^{j_{11}} C_{\pi(r_1)}(s_{\pi(r_1)}) \cdots$$
$$C_{\pi(1)}(s_{\pi(1)})\phi(\mu_{1k}^{q_{11}} \times \cdots \times \phi\mu_{nk}^{q_{n1}})(ds_1, \cdots, ds_{q_{11}+q_{21}+\cdots+q_{n1}})$$
$$- \int_{\Delta_{q_{11}+q_{21}+\cdots+q_{n1};r_1,\cdots,r_{h+1}}(\pi)}$$
$$C_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}) \cdots$$
$$C_{\pi(r_1+\cdots+r_h+1)}(s_{\pi(r_1+\cdots+r_h+1)})[p_{nh} A_n(\tau_h)]^{j_{nh}} \cdots [p_{2h} A_2(\tau_h)]^{j_{2h}}$$

$$[p_{1h}A_1(\tau_h)]^{j_{1h}}C_{\pi(r_1+\cdots+r_h)}(s_{\pi(r_1+\cdots+r_h)})\cdots C_{\pi(r_1+1)}(s_{\pi(r_1+1)})$$
$$[p_{n1}A_n(\tau_1)]^{j_{n1}}\cdots[p_{21}A_2(\tau_1)]^{j_{21}}[p_{11}A_1(\tau_1)]^{j_{11}}C_{\pi(r_1)}(s_{\pi(r_1)})\cdots$$
$$C_{\pi(1)}(s_{\pi(1)})\phi(\mu_1^{q_{11}}\times\cdots\times\mu_n^{q_{n1}})(ds_1,\cdots,ds_{q_{11}+q_{21}+\cdots+q_{n1}})||.$$

For each $l=1,\cdots,n, i=1,\cdots,h$, $p_{li}^k\to p_{li}$ and $\tau_{ik}\to\tau_i$ as $k\to\infty$. Hence since A_l is continuous we have

$$p_{li}^kA_l(\tau_{ik})\to p_{li}A_l(\tau_i)$$

as $k\to\infty$. Therefore, we have, for any $\phi\in X$

$$C_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\cdots+q_{n1})})\cdots.$$
$$C_{\pi(r_1+\cdots+r_h+1)}(s_{\pi(r_1+\cdots+r_h+1)})[p_{nh}^kA_n(\tau_{hk})]^{j_{nh}}\cdots[p_{2h}^kA_2(\tau_{hk})]^{j_{2h}}$$
$$[p_{1h}^kA_1(\tau_{hk})]^{j_{1h}}C_{\pi(r_1+\cdots+r_h)}(s_{\pi(r_1+\cdots+r_h)})\cdots C_{\pi(r_1+1)}(s_{\pi(r_1+1)})$$
$$[p_{n1}^kA_n(\tau_{1k})]^{j_{n1}}\cdots[p_{21}^kA_2(\tau_{1k})]^{j_{21}}[p_{11}^kA_1(\tau_{1k})]^{j_{11}}C_{\pi(r_1)}(s_{\pi(r_1)})\cdots$$
$$C_{\pi(1)}(s_{\pi(1)})\phi \qquad \to$$
$$C_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\cdots+q_{n1})})\cdots$$
$$C_{\pi(r_1+\cdots+r_h+1)}(s_{\pi(r_1+\cdots+r_h+1)})[p_{nh}A_n(\tau_h)]^{j_{nh}}\cdots[p_{2h}A_2(\tau_h)]^{j_{2h}}$$
$$[p_{1h}A_1(\tau_h)]^{j_{1h}}C_{\pi(r_1+\cdots+r_h)}(s_{\pi(r_1+\cdots+r_h)})\cdots C_{\pi(r_1+1)}(s_{\pi(r_1+1)})$$
$$[p_{n1}A_n(\tau_1)]^{j_{n1}}\cdots[p_{21}A_2(\tau_1)]^{j_{21}}[p_{11}A_1(\tau_1)]^{j_{11}}C_{\pi(r_1)}(s_{\pi(r_1)})\cdots$$
$$C_{\pi(1)}(s_{\pi(1)})\phi.$$

uniformly on $[0,T]^{q_{11}+\cdots+q_{n1}}$. $\{\mu_{1k}^{q_{11}}\times\cdots\times\mu_{nk}^{q_{n1}}\}$ is a sequence of continuous probability measures on $[0,T]^{q_{11}+\cdots+q_{n1}}$ since each term in the product is a continuous probability measure. And $[0,T]^{q_{11}+\cdots+q_{n1}}$ is separable. By Theorem 3.2 of [1] $\mu_{1k}^{q_{11}}\times\cdots\times\mu_{nk}^{q_{n1}}\rightharpoonup\mu_1^{q_{11}}\times\cdots\times\mu_n^{q_{n1}}$ since $\mu_{ik}\rightharpoonup\mu_i$ for each i. Hence we have, using Lemma 2.2 ,

$$\lim_{k\to\infty}\int_{\Delta_{q_{11}+q_{21}+\cdots+q_{n1};r_1,\cdots,r_{h+1}}(\pi)}$$
$$C_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\cdots+q_{n1})})\cdots$$
$$C_{\pi(r_1+\cdots+r_h+1)}(s_{\pi(r_1+\cdots+r_h+1)})[p_{nh}^kA_n(\tau_{hk})]^{j_{nh}}\cdots[p_{2h}^kA_2(\tau_{hk})]^{j_{2h}}$$
$$[p_{1h}^kA_1(\tau_{hk})]^{j_{1h}}C_{\pi(r_1+\cdots+r_h)}(s_{\pi(r_1+\cdots+r_h)})\cdots C_{\pi(r_1+1)}(s_{\pi(r_1+1)})$$
$$[p_{n1}^kA_n(\tau_{1k})]^{j_{n1}}\cdots[p_{21}^kA_2(\tau_{1k})]^{j_{21}}[p_{11}^kA_1(\tau_{1k})]^{j_{11}}C_{\pi(r_1)}(s_{\pi(r_1)})\cdots$$
$$C_{\pi(1)}(s_{\pi(1)})\phi(\mu_{1k}^{q_{11}}\times\cdots\times\mu_{nk}^{q_{n1}})(ds_1,\cdots,ds_{q_{11}+q_{21}+\cdots+q_{n1}})$$

$$= \int_{\Delta_{q_{11}+q_{21}+\cdots+q_{n1};r_1,\cdots,r_{h+1}}(\pi)} C_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\cdots+q_{n1})})\cdots C_{\pi(r_1+\cdots+r_h+1)}(s_{\pi(r_1+\cdots+r_h+1)})[p_{nh}A_n(\tau_h)]^{j_{nh}}\cdots[p_{2h}A_2(\tau_h)]^{j_{2h}} [p_{1h}A_1(\tau_h)]^{j_{1h}}C_{\pi(r_1+\cdots+r_h)}(s_{\pi(r_1+\cdots+r_h)})\cdots C_{\pi(r_1+1)}(s_{\pi(r_1+1)}) [p_{n1}A_n(\tau_1)]^{j_{n1}}\cdots[p_{21}A_2(\tau_1)]^{j_{21}}[p_{11}A_1(\tau_1)]^{j_{11}}C_{\pi(r_1)}(s_{\pi(r_1)})\cdots C_{\pi(1)}(s_{\pi(1)})\phi(\mu_1^{q_{11}}\times\cdots\times\mu_n^{q_{n1}})(ds_1,\cdots,ds_{q_{11}+q_{21}+\cdots+q_{n1}}).$$

Hence the conclusion follows. □

The following results can be obtained easily.

LEMMA 2.4. *Let $\lambda_1,\cdots,\lambda_n,\lambda_{1k},\cdots,\lambda_{nk}$, $k=1,2,\cdots$ be finite Borel measures. Suppose for $l=1,2,\cdots,n$*

$$\bar{r}_l = \sup\{r_l, r_{l1},\cdots,r_{lk},\cdots\} < \infty$$

where $r_l=\int_{[0,T]}||A_l(s)||\,|\lambda_l|(ds)$ and $r_{lk}=\int_{[0,T]}||A_l(s)||\,|\lambda_{lk}|(ds)$. Then for any $f\in\mathbb{A}\,(\bar{r}_1,\cdots,\bar{r}_n)$, $f((A_1(\cdot),\lambda_1)^{\sim},\cdots,(A_n(\cdot),\lambda_n)^{\sim})\in\mathbb{D}((A_1(\cdot),\lambda_1)^{\sim},\cdots,(A_n(\cdot),\lambda_n)^{\sim})$ and $f((A_1(\cdot),\lambda_{1k})^{\sim},\cdots,(A_n(\cdot),\lambda_{nk})^{\sim})\in\mathbb{D}((A_1(\cdot),\lambda_{1k})^{\sim},\cdots,(A_n(\cdot),\lambda_{nk})^{\sim})$ for any $k=1,2,\cdots$.

THEOREM 2.5. *Let the hypotheses of Theorem 2.3 be satisfied. Further suppose that for each $l=1,2,\cdots,n$ and $k=1,2,\cdots$, $\bar{r}_l, r_l, r_{lk}$ are given as in Lemma 2.4. Let $\mathcal{T}_{\lambda_{1k},\cdots,\lambda_{nk}}$ denote the disentangling map corresponding to the k^{th} term of sequences of measures. Then for any $f\in\mathbb{A}\,(\bar{r}_1,\cdots,\bar{r}_n)$ and for any $\phi\in X$*

$$\lim_{k\to\infty}\mathcal{T}_{\lambda_{1k},\cdots,\lambda_{nk}}f(A_1(\cdot)^{\sim},\cdots,A_n(\cdot)^{\sim})\phi = \mathcal{T}_{\lambda_1,\cdots,\lambda_n}f(A_1(\cdot)^{\sim},\cdots,A_n(\cdot)^{\sim})\phi.$$

Proof. We have

$$||\mathcal{T}_{\lambda_{1k},\cdots,\lambda_{nk}}f((A_1(\cdot),\lambda_{1k})^{\sim},\cdots,(A_n(\cdot),\lambda_{nk})^{\sim})\phi - \mathcal{T}_{\lambda_1,\cdots,\lambda_n}f((A_1(\cdot),\lambda_1)^{\sim},\cdots,(A_n(\cdot),\lambda_n)^{\sim})\phi||$$

$$\le \sum_{m_1,\cdots,m_n=0}^{\infty}|c_{m_1,\cdots,m_n}|\,||P^{m_1,\cdots,m_n}_{\lambda_{1k},\cdots,\lambda_{nk}}(A_1(\cdot),\cdots,A_n(\cdot))\phi - P^{m_1,\cdots,m_n}_{\lambda_1,\cdots,\lambda_n}(A_1(\cdot),\cdots,A_n(\cdot))\phi||$$

Note that

$$\begin{aligned}
&\sum_{m_1,\cdots,m_n=0}^{\infty} |c_{m_1,\cdots,m_n}|\, ||P^{m_1,\cdots,m_n}_{\lambda_{1k},\cdots,\lambda_{nk}}(A_1(\cdot),\cdots,A_n(\cdot))\phi \\
&\quad -P^{m_1,\cdots,m_n}_{\lambda_1,\cdots,\lambda_n}(A_1(\cdot),\cdots,A_n(\cdot))\phi|| \\
&\le \sum_{m_1,\cdots,m_n=0}^{\infty} |c_{m_1,\cdots,m_n}|\, ||\phi||\, \big[||P^{m_1,\cdots,m_n}_{\lambda_{1k},\cdots,\lambda_{nk}}(A_1(\cdot),\cdots,A_n(\cdot))|| \\
&\quad +||P^{m_1,\cdots,m_n}_{\lambda_1,\cdots,\lambda_n}(A_1(\cdot),\cdots,A_n(\cdot))||\big] \\
&\le ||\phi|| \sum_{m_1,\cdots,m_n=0}^{\infty} |c_{m_1,\cdots,m_n}|\, \big[[\int_{[0,T]} ||A_1(s)||\, |\lambda_{1k}|(ds)]^{m_1}\cdots \\
&\quad [\int_{[0,T]} ||A_n(s)||\, |\lambda_{nk}|(ds)]^{m_n} + [\int_{[0,T]} ||A_1(s)||\, |\lambda_1|(ds)]^{m_1}\cdots \\
&\quad [\int_{[0,T]} ||A_n(s)||\, |\lambda_n|(ds)]^{m_n}\big] \\
&= ||\phi|| \sum_{m_1,\cdots,m_n=0}^{\infty} |c_{m_1,\cdots,m_n}|\, \big[r_{1k}^{m_1}\cdots r_{nk}^{m_n} + r_1^{m_1}\cdots r_n^{m_n}\big] \\
&\le 2||\phi|| \sum_{m_1,\cdots,m_n=0}^{\infty} |c_{m_1,\cdots,m_n}|\, \bar{r}_1^{m_1}\cdots \bar{r}_n^{m_n}.
\end{aligned}$$

Since $\sum_{m_1,\cdots,m_n=0}^{\infty} |c_{m_1,\cdots,m_n}|\bar{r}_1^{m_1}\cdots\bar{r}_n^{m_n} < \infty$, by Theorem 1 and Lebesgue Dominated Convergence Theorem, we obtain a result. □

THEOREM 2.6. *Let the hypotheses of Theorem 1 be satisfied. Further assume that* $M_l = \sup_{s\in[0.T]} ||A_l(s)|| < \infty$ *for each* $l = 1,\cdots,n$. *Then for any* $f \in \mathbb{A}\ (2M_1,\cdots,2M_n)$ *and for any* $\phi \in X$

$$\begin{aligned}
&\lim_{k\to\infty} \mathcal{T}_{\lambda_{1k},\cdots,\lambda_{nk}} f(A_1(\cdot)\tilde{\ },\cdots,A_n(\cdot)\tilde{\ })\phi \\
&= \mathcal{T}_{\lambda_1,\cdots,\lambda_n} f(A_1(\cdot)\tilde{\ },\cdots,A_n(\cdot)\tilde{\ })\phi.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&||\mathcal{T}_{\lambda_{1k},\cdots,\lambda_{nk}} f((A_1(\cdot),\lambda_{1k})\tilde{\ },\cdots,(A_n(\cdot),\lambda_{nk})\tilde{\ })\phi \\
&-\mathcal{T}_{\lambda_1,\cdots,\lambda_n} f((A_1(\cdot),\lambda_1)\tilde{\ },\cdots,(A_n(\cdot),\lambda_n)\tilde{\ })\phi||
\end{aligned}$$

$$\leq \sum_{m_1,\cdots,m_n=0}^{\infty} |c_{m_1,\cdots,m_n}| \, ||P^{m_1,\cdots,m_n}_{\lambda_{1k},\cdots,\lambda_{nk}}(A_1(\cdot),\cdots,A_n(\cdot))\phi - P^{m_1,\cdots,m_n}_{\lambda_1,\cdots,\lambda_n}(A_1(\cdot),\cdots,A_n(\cdot))\phi||$$

Now

$$\begin{aligned}
&\sum_{m_1,\cdots,m_n=0}^{\infty} |c_{m_1,\cdots,m_n}| \, ||P^{m_1,\cdots,m_n}_{\lambda_{1k},\cdots,\lambda_{nk}}(A_1(\cdot),\cdots,A_n(\cdot))\phi \\
&\quad - P^{m_1,\cdots,m_n}_{\lambda_1,\cdots,\lambda_n}(A_1(\cdot),\cdots,A_n(\cdot))\phi|| \\
&\leq \sum_{m_1,\cdots,m_n=0}^{\infty} |c_{m_1,\cdots,m_n}| \, ||\phi|| \, \big[||P^{m_1,\cdots,m_n}_{\lambda_{1k},\cdots,\lambda_{nk}}(A_1(\cdot),\cdots,A_n(\cdot))|| \\
&\quad + ||P^{m_1,\cdots,m_n}_{\lambda_1,\cdots,\lambda_n}(A_1(\cdot),\cdots,A_n(\cdot))||\big] \\
&\leq ||\phi|| \sum_{m_1,\cdots,m_n=0}^{\infty} |c_{m_1,\cdots,m_n}| \, \big[[\int_{[0,T]} ||A_1(s)|| \, |\mu_{1k}|(ds) \\
&\quad + \sum_{i=1}^{h} p_{1i}||A_1(\tau_i)||]^{m_1} \cdots [\int_{[0,T]} ||A_n(s)|| \, |\mu_{nk}|(ds) \\
&\quad + \sum_{i=1}^{h} p^k_{ni}||A_n(\tau_{ik})||]^{m_n} + [\int_{[0,T]} ||A_1(s)|| \, |\mu_1|(ds) \\
&\quad + \sum_{i=1}^{h} p^k_{ni}||A_1(\tau_{ik})||]^{m_1} \cdots [\int_{[0,T]} ||A_n(s)|| \, |mu_n|(ds) \\
&\quad + \sum_{i=1}^{h} p_{ni}||A_n(\tau_i)||]^{m_n}\big] \\
&= ||\phi|| \sum_{m_1,\cdots,m_n=0}^{\infty} |c_{m_1,\cdots,m_n}| \, \big[(2M_1)^{m_1}\cdots(2M_n)^{m_n} \\
&\quad + (2M_1)^{m_1}\cdots(2M_n)^{m_n}\big] \\
&\leq 2||\phi|| \sum_{m_1,\cdots,m_n=0}^{\infty} |c_{m_1,\cdots,m_n}| \, (2M_1)^{m_1}\cdots(2M_n)^{m_n}.
\end{aligned}$$

Since $\sum_{m_1,\cdots,m_n=0}^{\infty} |c_{m_1,\cdots,m_n}|(2M_1)^{m_1}\cdots(2M_n)^{m_n} < \infty$, by Theorem 2.3 and Lebesgue Dominated Convergence Theorem, we obtain a result. □

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